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APPLICATION OF THE ENTROPY METHOD TO INVESTIGATION OF TRANSONIC ADIABATIC FLOWS

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Аннотация—Излагается энтропийный метод определения диссипации энергии в потоке движущейся жидкости. Дается методика расчёта трансзвукового адиабатического течения. На основании сопоставления расчётных результатов и данных опыта показывается, что предпосылки, лежащие в основе теории, хорошо подтверждаются в трансзвуковой области течения. Отмечается и обсуждается эффект затухания турбулентности при переходе через скорость звука. Излагается методика определения толщины вытеснения по данным одномерных продувок.

NOMENCLATURE

a,	velocity of sound;
e,	base of natural logarithm;
F,	cross sectional area of channel;
$f = \frac{F}{F_*}$,	dimensionless cross sectional channel area;
k,	index of an adiabatic line;
$K = \left(\frac{k+1}{2}\right)^{1/(k-1)}$,	dimensionless factor;
l_f,	friction work;
p,	pressure;
s,	entropy;
w,	velocity;
x,	co-ordinate;
$\bar{x}$,	dimensionless co-ordinate.

Greek symbols

δ^*,	attenuation thickness;
$\lambda = \frac{w}{a_*}$,	dimensionless velocity;
$\mu = \frac{d\sigma}{d\bar{x}}$,	coefficient;
ρ,	density;
$\tau = \frac{T}{T_*}$,	dimensionless temperature;
$\sigma = \frac{s}{R}$,	dimensionless entropy;
$\psi = f\lambda\tau^{1/(k-1)}$,	dimensionless function.

Suffixes

s,	isentropic process;
$*$,	critical cross section;
0,	initial state.

1. PRELIMINARY REMARKS

THE internal problem in gas dynamics, in general, consists in determining the laws of change in parameters of the medium moving along a channel under the influence of external forces. This problem is a very complicated one. Even in the simplest case of the stationary process when the energy exchange is absent (stationary adiabatic flow along a channel with fixed walls) the analytical investigation involves some difficulties which are almost insuperable. Under conditions investigated, the problem in principle reduces to establishing consistency between the spatial change of parameters of a medium and the known geometry of the channel. However, the situation is complicated to the greatest degree by the fact that this consistency does not reveal itself distinctly since the geometric effect is superimposed by dissipative effects, the intensity of which depends on the state of the moving medium. A fixed wall is a primary source of energy dissipation. The disturbances propagate into the flow due to an appropriate mechanism conditioned by the internal interaction of the elements of a moving liquid. A very complicated spatial pattern of flow results. This is the object of the present study. Considerable simplification may be achieved by substitution of this pattern by a one-dimensional scheme. The one-dimensional model of flow, the nature of which is connected with the idea of localization of all the dissipative effects directly on the surface of a wall, deviates considerably from real conditions of a flow. However, as is known, one-dimensional representation turns out to be very rewarding in investigations of many internal gas dynamic problems.

Within the limits of one-dimensional problems the intensity of dissipative effects is described in summary form in terms of a specific quantity namely the coefficient of resistance (or the coefficient of friction with which it is closely tied). Physically, this quantity is a measure of the ratio of energy, dissipated over an elementary portion of a channel (and related to dimensionless unit of its length), to the kinetic energy of the moving medium in the given section. The resistance coefficient is marked for the great simplicity of both its concept and of the character of operations, when applied to constant density fluids. As a characteristic of a gas dynamic flow, however, this quantity is expressed through the concept which is not free from unfavourable conditions which lead to fairly complicated experimental and numerical procedures. Nevertheless, the resistance coefficient is widely used as one of the basic values in theoretical investigations and engineering calculations dealing with a liquid motion of variable density. If only adiabatic flows are taken into account, and consequently, the rôle played by the resistance coefficient in the modern theory of convective heat transfer is not considered, the main reason why this concept has endured should be sought in the fact that for a channel with the given surface roughness the resistance coefficient may be taken as a value depending on the Reynolds number alone (at least, at velocities of flow not greatly exceeding the velocity of sound). As a matter of fact, it means that the influence of compressibility on the intensity of energy dissipation may be neglected. Formally it means that one may take into consideration a constant value of the resistance coefficient, since it is a weak function of the Reynolds number, varying but slightly over the length of a channel. It is easy to see the advantages gained due to this circumstance. However, at present one may consider it to be quite established that at flow velocities close to sonic the resistance coefficients start to decrease noticeably [1, 2]. Such changes are very marked directly in the transonic region. In any case we deal here with effects which cannot be neglected.

Thus, when studying near-sonic flow regions one has to face new circumstances which cast doubt on the expediency of preserving the system of investigation based on the concept of the resistance coefficient, other more effective methods of approach to energy dissipation must be evolved. When considering the possibilities which arise in this case, we take as a starting-point the notion that in some way or other it is expedient to associate the problem studied with the entropy concept, a single moving-medium parameter of state, change of which may be related to the amount of dissipated energy (since energy dissipation is a single, physically possible reason for entropy change under the conditions of an adiabatic process). This approach is based on very general concepts. However, one may indeed go further by considering the flows investigated to have velocities which differ but slightly from sonic. A quite different picture of entropy change, on the one hand, and of change in other moving-medium parameters, on the other, over the length of a channel, is a highly characteristic feature for this region of flow. Indeed, a sharp increase in the intensity with which all the parameters (velocity, pressure, temperature and density) change over the length is one of the main effects arising as sonic velocity is approached. Entropy is an exception, as it continues to increase at constant rate along a flow. Thus, within the region of flows which interested us, an entropy change along the axis of a channel is slow as compared to the change of other state parameters. This peculiarity in behaviour of entropy leads to the idea that the actual entropy change over a given length may be replaced by a linear approximation. This is quite advantageous. We use this approach as the basis of a new quantitative method of investigation of an energy dissipation process.

Natural as it is, the idea of a linear law of entropy change over the length is, of course, a special hypothesis ("linearity hypothesis") and must be verified experimentally.

2. ENTROPY METHOD

The proposed method of studying energy dissipation in a moving medium, which is referred to as "the entropy method", is considered in detail in [2]. Here we shall confine ourselves, therefore, to a short account of the principles.

Under the conditions of an adiabatic flow the amount of dissipated energy may be presented as

$$\mathrm{d}l_f = T\,\mathrm{d}s \qquad (1)$$

where $\mathrm{d}l_f$ is the friction work on an elementary portion related to a mass unit of a moving medium, $\mathrm{d}s$ is the corresponding entropy change, T is the thermodynamic temperature.

Let us reduce equation (1) to dimensionless form. And it will be considered that the equations valid for an ideal-gas state may be applied to a moving medium. We obtain

$$\frac{\mathrm{d}l_f}{RT_*} = \tau\,\mathrm{d}\sigma \qquad (1')$$

where T_* is the temperature at a critical section,

$$\tau = \frac{T}{T_*} = \frac{k+1}{2}\left(1 - \frac{k-1}{k+1}\right)\lambda^2$$

is the dimensionless temperature, a_* is the critical velocity, R is the gas constant.

Note, that here the dimensionless temperature is related to the temperature in the critical condition. It differs from a normally applied dimensionless temperature (related to the stagnation temperature) by the factor $(k+1)/2$.

Now we may express $\mathrm{d}\sigma$ in terms of dimensionless variables [1] and get

$$\mathrm{d}\sigma = \frac{1}{k-1}\,\frac{\mathrm{d}T}{T} + \frac{\mathrm{d}v}{v},$$

hence, due to the law of constant mass flow,

$$\mathrm{d}\sigma = \frac{1}{k-1}\,\frac{\mathrm{d}T}{T} + \frac{\mathrm{d}w}{w} + \frac{\mathrm{d}F}{F}.$$

In dimensionless parameters this equation may be arranged to give

$$\mathrm{d}\sigma = \frac{1}{k-1}\,\frac{\mathrm{d}\tau}{\tau} + \frac{\mathrm{d}\lambda}{\lambda} + \frac{\mathrm{d}f}{f},$$

or

$$\mathrm{d}\sigma = \mathrm{d}\ln(f\lambda\tau^{1/(k-1)}). \qquad (2)$$

Denote

$$\psi \equiv f\lambda\tau^{1/(k-1)}. \qquad (3)$$

Then

$$\mathrm{d}\sigma = \mathrm{d}\ln\psi = \frac{\mathrm{d}\psi}{\psi}. \qquad (4)$$

Apparently, it is in many respects more convenient to use the value ψ than σ. As it follows directly from equation (3) at a critical section the value ψ (as well as the allied factors f, λ, τ forming it) reduces to unity.

The essential advantage of the system of reduced parameters lies in the fact that they are interconnected as single valent functions and therefore, the prediction of one determines all of them.

In particular, expressing τ in terms of λ, we obtain

$$\psi = Kf\lambda\left(1 - \frac{k-1}{k+1}\lambda^2\right)^{1/(k-1)} \qquad (6)$$

where

$$K = \left(\frac{k+1}{2}\right)^{1/(k-1)}.$$

Now on the basis of the linearity hypothesis we introduce a special value μ by the equation

$$\mu = \frac{\mathrm{d}\sigma}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}\ln\psi}{\mathrm{d}\bar{x}} = \text{const.} \qquad (7)$$

For the entropy method the coefficient μ assumes the value of a fundamental characteristic of hydrodynamic properties of a channel.

Further relationships are obtained quite easily. Assuming $\bar{x} = 0$ for a critical section, we get from equation (7)

$$\ln\psi = \mu\bar{x},$$

or

$$\psi = e^{\mu\bar{x}}. \qquad (8)$$

It is useful to give this result in another form. Under the conditions of isentropic flow

$$\psi = \text{const.} = 1$$

and, correspondingly,

$$f = \frac{1}{\lambda \tau^{1/(k-1)}}.$$

Therefore

$$\lambda \tau^{1/(k-1)} = \frac{1}{f_s}.$$

Note that the value $1/f_s$ for the given λ equals the corresponding value of the gas dynamic function q.

Substituting this expression into equation (3), we get

$$\psi = \frac{f}{f_s}$$

Now equation (7) is reduced to the form

$$f_s = e^{-\mu \bar{x}} f. \tag{9}$$

Equation (9) refers to the relationship between areas (equation of areas) determining the flow expansion ratio under the conditions of a real and equivalent flow (for which the velocity curves $\lambda = f(\bar{x})$ are identical). However, one should be aware that at the same time the value f_s is a certain gas dynamic function which is in a mutual single valent relationship with the reduced medium parameters

$$\left[\text{e.g. } f_s = \frac{l}{K\lambda\left(1 - \dfrac{k-1}{k+1}\lambda^2\right)^{1/(k-1)}}\right].$$

The real meaning of the equation consists, therefore, in establishing the law for the variation in flow over the length of channel ($\bar{x}$) for a given profile (the form of the function $f = f(\bar{x})$ is assumed to be known from the boundary conditions). The problem is solved. Equation (9), however, contains the coefficient μ which serves as a constant parameter and should be predicted. This question needs special consideration.

3. SOME EXPERIMENTAL RESULTS

The coefficient μ is determined by equation (7) in the form:

$$\mu = -\frac{d \ln \psi}{d\bar{x}}$$

Thus, the problem reduces to determination of the form of the function $\psi(\bar{x})$. It is immediately seen from equation (6) that distribution of the function ψ along the axis is conditioned by the laws of a conjugated change of the cross section and parameters of the flow over the length of a channel (for the character of the relation between change in flow parameters and that of a channel profile is dependent on the development of the energy dissipation process). Consequently, in each particular case the determination of the form of the function $\psi = \psi(\bar{x})$ reduces to establishing the relation between $\lambda(\bar{x})$ and $f(\bar{x})$. Such a co-ordination of the values λ and f may be realized on the basis of experimental data which at present can be carried out with adequate accuracy [1, 2, 3].

The method applied is based on the static pressure (p) distribution over the length of a channel and, consequently, the experimental part consists in determining the shape of the pressure curve. Appropriate measurements are taken with the help of a "string-type" probe stretched along the axis of a channel [1]. Experimental data available at present allow some reliable conclusions to be drawn and, in the first place, assessment to be made of the reliability of the linearity hypothesis and of the regions to which it applies.

In Figs. 1 and 2, diagrams which were the first to illustrate the validity of the linearity hypothesis are plotted. They represent a very descriptive form of verification of the fact that the law of a velocity change over the length of a channel, obtained from calculation with the assumption of the constant coefficient μ, agrees with the experimental curve. The scheme of plotting consists of the following. In the $\lambda - \bar{x}$ plane the results of calculation are given in the form of a family of curves with the parameter μ. The curves corresponding to monotonically decreasing values of μ are located one above the other. The curve of the isentropic state change corresponding to the value $\mu = 0$ makes up the upper border of the family. Experimental points are plotted on this very plane.

Figure 1 represents the results obtained from an investigation of a near-sonic flow in an exit portion of a tube with a constant cross section. Note that under these conditions, despite the absence of geometric effects, a change of parameters along the axis of a channel occurs already

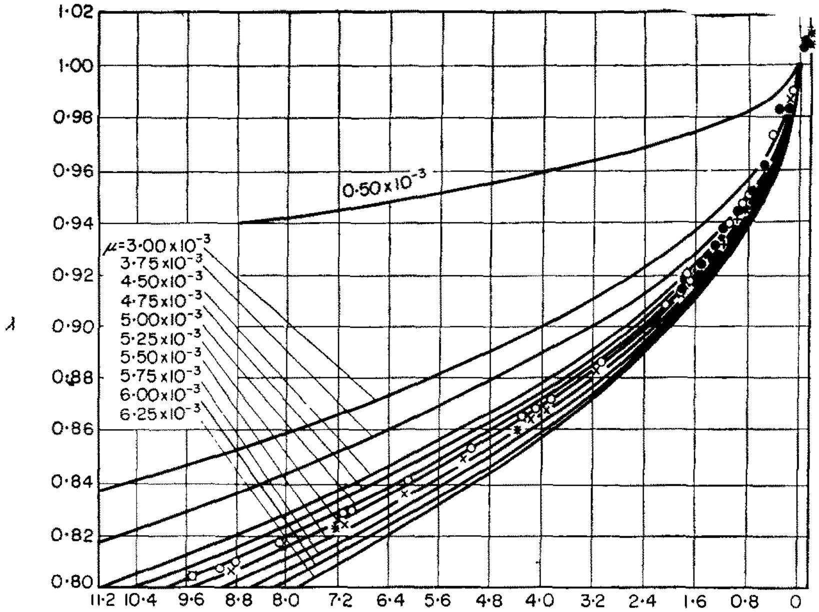

FIG. 1. Analytical curves and experimental points for $\lambda = f(\bar{x})$ at various values of μ at the exit portion of a cylindrical tube.
● ○ ◇ regime 4, *Re* = 7·00 × 10⁵; *G* = 0·249 kg/s
× * regime 9, *Re* = 6·25 × 10⁵; *G* = 0·222 kg/s

sufficiently rapidly. Therefore, the considerations which lie in the basis of the linearity hypothesis hold true. The relations are simpler in the case of a channel of a constant cross section. Apparently, in this case we must assume that $f = 1$ and correspondingly, $\psi = \lambda \tau^{1/(k-1)}$. The line $\lambda = 1$ is the upper boundary of the family. It is distinctly seen from the charts that, despite the density of the network (the interval for μ is about 5 per cent), experimental points are located on narrow strips bounded by adjacent curves over a considerable range right up to values of λ of the order 0·92.

Figure 2 refers to the case of a supersonic flow in an axisymmetric channel of a small conicity. It is easy to see that in this case theoretical and experimental results agree well between themselves. Experimental points give a quite distinct curve which over its whole length can be referred to the same family (obtained by calculation), the parameter μ being of the value close to 0·014. Further experiments on supersonic nozzles of small (but different) conicity convincingly proved the assumption that the coefficient μ is constant for the whole investigated range of λ (up to the values of λ of the order 1·5). The obtained values to, of μ lie within the comparatively short interval from 0·012 to 0·018.

The region close to the velocity of sound (e.g. from $\lambda = 0·92$) requires special consideration. It is easy to see that in this case experimental points have a steeper initial gradient than the analytical curves. Such a distribution of experimental points should be, apparently, attributed to the fact, that the intensity of energy dissipation decreases under the present conditions occuring near the critical state. No other explanations of the studied peculiarity in the position of experimental points are admissible since the coefficient μ is directly related to the intensity of dissipation. Therefore, in contrast to the considerations on the possible causes of the sharp decrease in the resistance coefficient [1], we state here that the decrease in the coefficient μ is caused by some real physical effect that results in weakening of dissipation mechanism. The peculiarity in behaviour of the coefficient μ

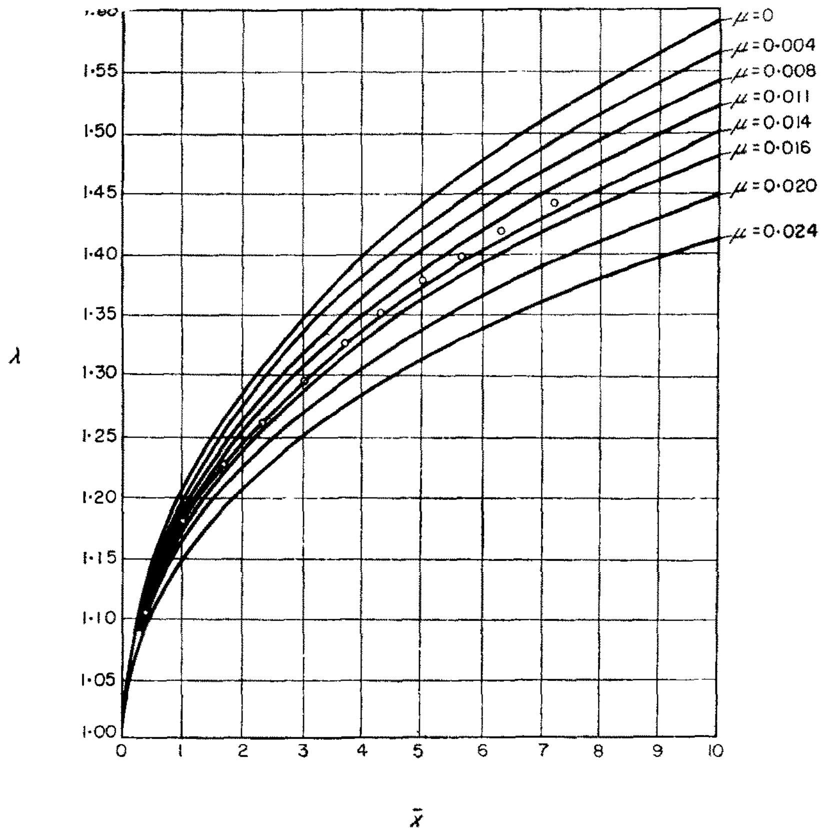

FIG. 2. Analytical curves and experimental points for $\lambda = f(\bar{x})$ at various values of μ in a channel of a variable cross section (conicity angle = 1°27′).

makes us think of turbulence attenuation as a possible source peculiar to the hydrodynamic situation of a transonic region. This idea is confirmed by some other experimental facts observed in flows with velocities highly close to the velocity of sound (such as the unusual character of a flow round blunt bodies, laminarization of a velocity profile, change in velocity oscillograms). It is natural to relate the effect of attenuation to the initial cause, i.e. to the action of the considerable negative pressure gradients typical of near-critical states. Such an assumption as our idea of turbulence attenuation, touching upon the very basic properties of a flow and having extremely far-reaching implications, should be, of course, very thoroughly founded. With appropriate caution therefore, we advance it for the present as a preliminary hypothesis.

4. ENTROPY METHOD AND A TWO-DIMENSIONAL FLOW MODEL

The entropy method in its basic concept, appertains to a one-dimensional theory. This, however, does not exclude it from being applied to studying the problems occuring clearly outside those of a one-dimensional model. We will show that application of the entropy method makes it possible to find a very simple way for solving the main problem of the two-dimensional theory calculation of attenuation thickness [4].

We will deal with the case of a flow along a conical supersonic nozzle, i.e. along an axisymmetrical short channel (a flow with boundary layers not closed up), the attenuation thickness δ^* may be given, according to the definition, as

$$\delta^* = R - R' \tag{10}$$

where R and R' are radii of cross sections (F and

F') corresponding to the conditions of real and ideal (non-viscous) flow. These conditions are connected between themselves by the condition that the mass flow and the velocity distribution along the length of the axis and of static pressure should be identical. The problem is thus reduced to determination of R' for a given R. We may solve it easily with the help of the entropy method. And, as it is usually assumed, we shall neglect energy dissipation in the subsonic portion of the nozzle, i.e. we shall assume that a boundary layer starts growing from the critical cross section.

Compare three different flow patterns: real flow, ideal flow and equivalent isentropic flow. Under real flow conditions to each given section (i.e. to each section x or $\bar{x}$) there is assigned definite values of the parameters, which have been averaged in accordance with the single dimensional model: velocity $\bar{w}$ or $\bar{\lambda}$, temperature $\bar{T}$, $\bar{\tau}$, density $\bar{\rho}$, $\bar{\rho}/\rho_0$, and also those related to the potential flow region (isentropic) according to two dimensional representations: w (or λ), T (or τ), ρ (or (ρ/ρ_0)). For pressure, of course, only one value of p or p/p_0 is possible. Moreover, the area of a cross section $F = \pi R^2$ (or f) is introduced. With regard to two other cases (which we understand to be in strict accordance with the definitions given above) such a duality does not take place. An ideal flow is represented by the values of velocity $w'(\lambda')$, temperature $T'(\tau')$, density $\rho'(\rho'/\rho_0)$, pressure $p'(p'/p_0)$ and cross sectional area $F' = \pi R'^2(f')$. Correspondingly, for an isentropic flow we have the values $w_s(\lambda_s)$, $T_s(\tau_s)$, $\rho_s(\rho_s/\rho_0)$, $p_s(p_s/p_0)$ and $F_s(f_s)$. According to the definition we should have: $w' = w(\lambda' = \lambda)$; $T' = T(\tau' = \tau)$; $\rho' = \rho$; $p' = p$ and equally $w_s = \bar{w}(\lambda_s = \bar{\lambda})$; $T_s = \bar{T}(\tau_s = \bar{\tau})$. Moreover, on the basis of area equation (9) one gets $f_s = f\,e^{-\mu\bar{x}}$. Finally, all three cases are connected by the condition of identity of mass flow $G' = G_s = G$.

Now it is easy to see how the solution can proceed. To determine R' it is sufficient to find $\lambda' = \lambda$ since under the conditions of an ideal flow the reduced area (and then $F' = f'F_*$) is directly expressed in terms of the reduced velocity. And for the ideal flow the reduced velocity may be simply related to pressure since it follows from $T/T_0 = (p/p_0)^{(k-1)/k}$ that

$$\left(\frac{p}{p_0}\right)^{(k-1)/k} = 1 - \frac{k-1}{k+1}\lambda^2$$

or, finally,

$$\lambda = \sqrt{\left\{\frac{k+1}{k-1}\left[1 - \left(\frac{p}{p_0}\right)^{(k-1)/k}\right]\right\}}.$$

We see that the problem is reduced to the determination of static pressure under the conditions of a real flow. Here the following method of a solution is possible. The demand of identity of the rate ($G_s = G$) leads to the relation $F_p = F_s p_s$, hence

$$p = p_s \frac{F_s}{F} = p_s\,e^{-\mu\bar{x}} \tag{11}$$

However, p_s is expressed in terms of $\lambda_s = \bar{\lambda}$ as

$$p_s = p_0\left(1 - \frac{k-1}{k+1}\lambda_s^2\right)^{k/(k-1)}.$$

As to the value λ_s it may be found directly as a function of the reduced area in an isentropic flow $f_s = f\,e^{-\mu\bar{x}}$.

Thus, there is the following procedure of solution. First, we determine f_s by the given area of a real channel. Then, from the tables of gas dynamic functions we find $\lambda_s = \bar{\lambda}$ and p_s. Further, from equation (11) we obtain the real pressure $p = p'$ and the dimensionless velocity λ' under the conditions of an ideal flow (equal to the reduced velocity of a potential core of a flow). From the tables of gas dynamic functions we define f' and then F' and δ^* in accordance with the values of $\lambda' = \lambda$ obtained. We see that the solution of the problem is very simple. The problem is regarded, of course, as purely theoretical and reduces to the determination of the attenuation thickness $\delta^*(\bar{x})$ by the given constant value of μ. Any considerations on the physical conditions of the boundary layer formation and, in particular on the relation of flow regime in a boundary layer to the value μ, do not enter the scope of this problem. Note, that equation (11), being an analogue of area equation (9), is of interest in itself, since it allows one to plot an analytical curve of static pressure distribution along the axis of a channel.

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REFERENCES

1. A. F. GANDESLMAN, A. A. GUKHMAN, N. V. ILYUKHIN and L. N. NAURITS, Investigation of the resistance coefficient in a near-sonic flow (Issledovanie koeffitsienta soprotivleniya pri techenii s okolozvukovoi skorost'yu) *Zh. Tech. Fiz.* **24**, 12, 2234–2249 (1954).
2. A. A. GUKHMAN, A. F. GANDELSMAN and L. N. NAURITS, On hydrodynamic resistance in a transonic region of a flow (O gidrodinamicheskom soprotivlenii v transzvukovoi oblasti techeniya). *Energomashinostroenie* No. 7, 10–14 (1957).
3. A. F. GANDELSMAN, A. A. GUKHMAN and N. V. ILYUKHIN, Investigation of resistance coefficient variation in a supersonic gas flow (Issledovanie izmeneniya koeffitsienta soprotivleniya pri techenii gaza so sverkhzvukovoi skorostyu). *Teploenergetika* No. 1, 17–23 (1955).
4. A. A. GUKHMAN and A. F. GANDELSMAN, Application of an entropy method to the determination of attenuation thickness of an adiabatic flow in a supersonic nozzle (Primenenie entropiinogo metoda k opredeleniyu tolshchiny vytesneniya pri adiabaticheskom techenii v sverkhzvukovom sople). *Inzh. Fiz. Zh.* **4**, 9, 73–75 (1961).

Abstract—An entropy method is presented for determining energy dissipation in a moving fluid. Methods for calculation of a transonic adiabatic flow are given. Comparison between the theoretical and experimental results shows that the assumptions, being the basis of the theory, are well confirmed in a transonic flow region. The effect of turbulence attenuation during transition through the velocity of sound is noted and discussed. The methods for determining the thickness of the attenuation thickness according to data from single-dimensional blow-down testing are considered.

Résumé—Cet article présente une méthode "entropique" permettant de déterminer la dissipation d'énergie dans un fluide en écoulement. Des méthodes de calcul d'un écoulement transonique adiabatique sont données. La comparaison entre les résultats théoriques et expérimentaux montre que les hypothèses, qui sont à la base de la théorie, se confirment bien dans la région d'écoulement transonique. L'effet d'atténuation de la turbulence au cours de la transition sonique est mis en évidence et étudié. Les méthodes de détermination de l'épaisseur d'atténuation, à partir de données expérimentales relatives à un écoulement unidimensionnel sont analysées.

Zusammenfassung—Die Energieverteilung in einer bewegten Flüssigkeit wird mit einer Entropiemethode bestimmt. Berechnungsmethoden für die adiabate Strömung im Bereich der Schallgeschwindigkeit sind angegeben. Ein Vergleich der theoretischen mit experimentellen Ergebnissen gibt für die als Grundlage der Theorie gemachten Annahmen gute Übereinstimmung im Bereich der Schallgeschwindigkeit. Der Effekt der während des Durchgangs durch die Schallgeschwindigkeit festgestellten Turbulenzdämpfung wurde diskutiert. Die Methoden zur Ermittlung der Dämpfungsbreite aus Daten von eindimensionalen Strömungsversuchen sind berücksichtigt.